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BIALGEBRA STRUCTURE ON BRIDGELAND'S HALL ALGEBRA OF TWO-PERIODIC COMPLEXES

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Abstract. We study the bialgebra structure of the Hall algebra of two-periodic complexes recently introduced by Bridgeland. We introduce coproduct on Bridgeland's Hall algebra, and show that in the hereditary case the resulting bialgebra structure coincides with that on Drinfeld double of the ordinary Hall algebra

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0. Introduction

0.1. This paper is a sequel to our previous paper [Ya], where we studied the Hall algebra of $\mathbb{Z}/2\mathbb{Z}$-graded complexes introduced by Bridgeland [Br1]. For any abelian category $\mathcal{A}$ over a finite field $\mathfrak{k} := \mathbb{F}_q$ with finite dimensional morphism spaces, one can consider the Hall algebra $\mathcal{H}(\mathcal{A})$, which was introduced by Ringel [Ri], and its twisted version $\mathcal{H}_{\mathrm{tw}}(\mathcal{A})$. Let us denote by $\mathcal{P} \subset \mathcal{A}$ be the subcategory of projective objects. The algebra $\mathcal{DH}(\mathcal{A})$ Bridgeland introduced is a localization of $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$ by the set of acyclic complexes:

$$\mathcal{DH}(\mathcal{A}) := \mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))\big[[M_\bullet]^{-1} \mid H_*(M_\bullet) = 0\big]. \tag{0.1}$$

Here $\mathcal{C}(\mathcal{P}) \equiv \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{P})$ is the category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of objects in $\mathcal{P}$, which is a subcategory of the abelian category $\mathcal{C}(\mathcal{A}) \equiv \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{A})$ of $\mathbb{Z}/2\mathbb{Z}$-graded complexes of objects in $\mathcal{A}$. $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$ means a subalgebra of the twisted Hall algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{A}))$ generated by $\mathcal{C}(\mathcal{P})$.

In [Br1], it was shown that $\mathcal{DH}(\mathcal{A})$ is an associative algebra with unit, and that it has a nice basis if $\mathcal{A}$ is hereditary. As a result, Bridgeland was able to show [Br1, Theorem 1.2] that one has an embedding of algebras

$$U_t(\mathfrak{g}_Q) \hookrightarrow \mathcal{DH}_{\mathrm{red}}(\mathrm{Rep}_{\mathfrak{k}}(Q)).$$

Here $\mathcal{A} = \mathrm{Rep}_{\mathfrak{k}}(Q)$ is the category of finite-dimensional $\mathfrak{k}$-representations of a finite quiver Q without oriented cycles, and $\mathcal{DH}_{\mathrm{red}}(\mathcal{A})$ is a certain quotient of $\mathcal{DH}(\mathcal{A})$. $\mathfrak{g}_Q$ is the derived Kac-Moody Lie algebra associated to Q, and $U_t(\mathfrak{g}_Q)$ is the quantized enveloping algebra of $\mathfrak{g}_Q$ with $t = \sqrt{q}$ a fixed square root of $q = \#\mathfrak{k}$.

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The result (0.1) is a natural extension of the classical result due to Ringel [Ri] and Green [Gr], where the upper half (or the Borel part) of the whole quantum group is realized as a subalgebra of the ordinary (twisted) Hall algebra.

In the previous paper [Ya], we showed that for a general hereditary category $\mathcal{A}$ the algebra $\mathcal{DH}(\mathcal{A})$ is, as an associative algebra, isomorphic to the Drinfeld double [Dr] of the extended Hall bialgebra $\mathcal{H}^{\mathrm{e}}_{\mathrm{tw}}(\mathcal{A})$ as an associative algebra. Let us mention that this claim was stated without proof in [Br1, Theorem 1.2]. Combining it with the result of Cramer [1], one can prove the invariance of $\mathcal{DH}(\mathcal{A})$ under the derived equivalence of $\mathcal{A}$. Let us also mention that in a recent work of Gorsky [Go] more abstract approach is taken to prove the derived equivalence.

A natural question on these results is whether one can introduce a coproduct and a bialgebra structure on $\mathcal{DH}(\mathcal{A})$. In this paper we show that there exists a coproduct on $\mathcal{DH}(\mathcal{A})$ which gives a bialgebra structure. Our coproduct is, as a result, a natural analogue of that on the ordinary Hall algebra introduced by Green [Gr]. We recommend [Sc, § 1.5] and [R2, Part I, II] for detailed and nice reviews of Green's coproduct and the proof of its bialgebra property.

However, our coproduct is in some sense artificial. One of the obstructions of constructing coproduct is that Bridgeland's Hall algebra is defined as a localization of (non-commutative) algebra. Although one can define a coproduct on the unlocalized algebra by taking a straightforward analogue of Green's coproduct, it doesn't descend to the localized algebra. See §2.2 for a detailed explanation.

Our strategy to handle this obstruction is substituting the set of exact sequences in the category of complexes by a restricted one, and making the structure constants for coproduct smaller. To realize this idea, we use the notion of exact category in the sense of Quillen [Qu]. By Hubery [Hu] one can construct Hall algebra from an exact category, and the resulting algebra is indeed a unital associative algebra. We will introduce an exact category structure on the category of two-periodic complexes, and from the resulting Hall algebra for an exact category, we make a coassociative coproduct on Bridgeland's Hall algebra. See §2.3 and §2.4 for the detail. Let us mention that our strategy is inspired by the recent work [Go].

In §3 we treat the case where $\mathcal{A}$ is hereditary. In this case our coalgebra structure is compatible to the embedding $\mathcal{H}(\mathcal{A}) \hookrightarrow \mathcal{DA}(\mathcal{H})$ of the ordinary Hall algebra into Bridgeland's Hall algebra. In this sense, our coproduct is a natural analogue of Green's coproduct.

Let us close this introduction by indicating further directions. The first direction is investigating Hopf algebra structure. In [2], Xiao introduced a (topological) Hopf algebra structure on the ordinary Hall algebra. We expect a similar Hopf algebra structure can be introduced on the algebra $\mathcal{DA}(\mathcal{H})$. Another direction is the investigation of higher dimensional case. As mentioned in [Br1], $\mathcal{DA}(\mathcal{H})$ makes sense for abelian category of arbitrary global dimension, but it looks too large. Our construction of coproduct might suggest that in order to obtain a moderate algebra, it is better to consider a restriction on the counting of extensions. We expect that stability conditions on triangulated categories, which was also introduced by Bridgeland [Br2], is related to this direction.

0.2. **Notations and conventions.** Here we indicate several global notations.

$\mathfrak{k} := \mathbb{F}_q$ is a fixed finite field unless otherwise stated, and all the categories will be $\mathfrak{k}$-linear. We choose and fix a square root $t := \sqrt{q}$.

For an abelian category $\mathcal{A}$, we denote by $\mathrm{Obj}(\mathcal{A})$ the class of objects of $\mathcal{A}$. For an object M of $\mathcal{A}$, the class of M in the Grothendieck group $K(\mathcal{A})$ is denoted by $\widehat{M}$. The subcategory of $\mathcal{A}$ consisting of projective objects is denoted by $\mathcal{P}$.

In our argument we impose several assumptions on an abelian category $\mathcal{A}$. Let us introduce the following conditions:

(a) essentially (= skeletally) small with finite morphism spaces,
(b) linear over $\mathfrak{k}$,
(c) of finite global dimension and having enough projectives.
(d) $\mathcal{A}$ is hereditary, that is of global dimension at most 1,
(e) nonzero objects in $\mathcal{A}$ define nonzero classes in $K(\mathcal{A})$.

For an abelian category $\mathcal{A}$ which is essentially small, the set of its isomorphism classes is denoted by $\mathrm{Iso}(\mathcal{A})$.

For a complex $M_\bullet = (\cdots \to M_i \xrightarrow{d_i} M_{i+1} \to \cdots)$ in an abelian category $\mathcal{A}$, its homology is denoted by $H_*(M_\bullet)$.

For a set S, we denote by $\left|S\right|$ its cardinality.

1. Hall algebras of complexes

We summarize necessary definitions and properties of Hall algebras of $\mathbb{Z}/2\mathbb{Z}$-graded complexes. Most of the materials were introduced and shown in [Br1].

1.1. **$\mathbb{Z}/2\mathbb{Z}$-periodic complexes.** We will recall the basic definitions in [Br1, §3]. Let us fix an abelian category $\mathcal{A}$.

Let $\mathcal{C}(\mathcal{A}) \equiv \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{A})$ be the abelian category of $\mathbb{Z}/2\mathbb{Z}$-graded complexes in $\mathcal{A}$. An object $M_\bullet$ of $\mathcal{C}(\mathcal{A})$ consists of the following diagram in $\mathcal{A}$:

$$M_1 \underset{d_0^M}{\overset{d_1^M}{\rightleftarrows}} M_0 \ , \quad d_{i+1}^M \circ d_i^M = 0.$$

A morphism $s_\bullet : M_\bullet \to N_\bullet$ consists of a diagram

$$\begin{tikzcd} M_1 \arrow[r, shift left, "d_1^M"] \arrow[d, "s_1"'] & M_0 \arrow[l, shift left, "d_0^M"] \arrow[d, "s_0"] \\ N_1 \arrow[r, shift left, "d_1^N"] & N_0 \arrow[l, shift left, "d_0^N"] \end{tikzcd}$$

with $s_{i+1} \circ d_i^M = d_i^N \circ s_i$. Here and hereafter indices in the diagram of an object in $\mathcal{C}(\mathcal{A})$ are understood by modulo 2. We also denote by $0 \equiv 0_\bullet$ the trivial $\mathbb{Z}/2\mathbb{Z}$-graded complex (where the graded parts are equal to the zero object of $\mathcal{A}$).

Two morphisms $s_\bullet, t_\bullet : M_\bullet \to N_\bullet$ are said to be homotopic if there are morphisms $h_i : M_i \to N_{i+1}$ such that $t_i - s_i = d'_{i+1} \circ h_i + h_{i+1} \circ d_i$. Denote by $\mathrm{Ho}(\mathcal{A}) \equiv \mathrm{Ho}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{A})$ the category obtained from $\mathcal{C}(\mathcal{A})$ by identifying homotopic morphisms.

Let us also denote by $\mathcal{C}(\mathcal{P}) \equiv \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{P}) \subset \mathcal{C}(\mathcal{A})$ the full subcategory whose objects are complexes of projectives in $\mathcal{A}$.

For an object $M_\bullet$ of $\mathcal{C}(\mathcal{A})$, we define its class $\widehat{M_\bullet}$ in the Grothendieck group $K(\mathcal{A})$ to be

$$\widehat{M_\bullet} := \widehat{M_0} - \widehat{M_1} \in K(\mathcal{A}).$$

The shift functor [1] of complexes induces an involution $\mathcal{C}(\mathcal{A}) \overset{*}{\longleftrightarrow} \mathcal{C}(\mathcal{A})$. This involution shifts the grading and changes the sign of the differential as follows:

$$M_\bullet = \ M_1 \underset{d_0^M}{\overset{d_1^M}{\rightleftarrows}} M_0 \qquad \overset{*}{\longleftrightarrow} \qquad M_\bullet^* = \ M_0 \underset{-d_1^M}{\overset{-d_0^M}{\rightleftarrows}} M_1$$

1.2. **Bridgeland's Hall algebra of complexes.** Let us recall the definition of the ordinary Hall algebra. For an abelian category $\mathcal{A}$ satisfying the condition (a), consider the vector space

$$\mathcal{H}(\mathcal{A}) := \bigoplus_{A \in \mathrm{Iso}(\mathcal{A})} \mathbb{C}[A]$$

linearly spanned by symbols $[A]$ with A running through the set $\mathrm{Iso}(\mathcal{A})$ of isomorphism classes of objects in $\mathcal{A}$. Then, by Ringel [Ri], the following operation $\diamond$ defines on $\mathcal{H}(\mathcal{A})$ a structure of unital associative algebra over $\mathbb{C}$:

$$[A] \diamond [B] := \sum_{C \in \mathrm{Iso}(\mathcal{A})} \frac{\left|\mathrm{Ext}^1_{\mathcal{A}}(A,B)_C\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(A,B)\right|}[C].$$

Here

$$\mathrm{Ext}^1_{\mathcal{A}}(A,B)_C \subset \mathrm{Ext}^1_{\mathcal{A}}(A,B)$$

is the set parametrizing extensions of B by A with the middle term isomorphic to C. The class $[0]$ of the zero object is the unit for this product $\diamond$, and the algebra $(\mathcal{H}(\mathcal{A}), \diamond, [0])$ is called the Hall algebra of $\mathcal{A}$.

For an abelian category $\mathcal{A}$ satisfying the condition (a), Let $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ be the Hall algebra of the category $\mathcal{C}(\mathcal{P})$. As a $\mathbb{C}$-vector space we have $\mathcal{H}(\mathcal{C}(\mathcal{P})) = \bigoplus_{M_\bullet \in \mathrm{Iso}(\mathcal{C}(\mathcal{P}))} \mathbb{C}[M_\bullet]$, and the product is given by the formula

$$[M_\bullet] \diamond [N_\bullet] := \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C}(\mathcal{P}))} \frac{\left|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)_{L_\bullet}\right|}{\left|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)\right|}[L_\bullet].$$

Since the subcategory $\mathcal{C}(\mathcal{P}) \subset \mathcal{C}(\mathcal{A})$ is closed under extensions and since the space $\mathrm{Ext}^1_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)$ is always of finite dimension, this expression makes sense. Indeed, according to [Br1, Lemma 3.3] we have $\mathrm{Ext}^1_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{A})}(M_\bullet, N_\bullet^*)$.

Remark 1.1. As in [Br1, §2.3], using $[[M_\bullet]] := [M_\bullet]/\left|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(M_\bullet)\right|$ one can rewrite the multiplication as

$$[[M_\bullet]] \diamond [[N_\bullet]] = \sum_{L_\bullet} g^{L_\bullet}_{M_\bullet, N_\bullet} [[L_\bullet]]$$

with

$$g^{L_\bullet}_{M_\bullet, N_\bullet} := \left|\{N'_\bullet \subset L_\bullet \mid N'_\bullet \cong N_\bullet,\ L_\bullet/N_\bullet \cong M_\bullet\}\right| \tag{1.1}$$

for $L_\bullet, M_\bullet, N_\bullet \in \mathrm{Obj}(\mathcal{C}(\mathcal{A}))$. Here we used the well-known formula

$$g^A_{B,C} = \frac{\left|\mathrm{Ext}_{\mathcal{A}}(B,C)_A\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(B,C)\right|} \frac{\left|\mathrm{Aut}_{\mathcal{A}}(A)\right|}{\left|\mathrm{Aut}_{\mathcal{A}}(B)\right|\left|\mathrm{Aut}_{\mathcal{A}}(C)\right|} \tag{1.2}$$

for any abelian category $\mathcal{A}$ and its objects A, B, C. Then the associativity of $\diamond$, that is, $([M_\bullet] \diamond [N_\bullet]) \diamond [P_\bullet] = [M_\bullet] \diamond ([N_\bullet] \diamond [P_\bullet])$ for any $M_\bullet, N_\bullet, P_\bullet \in \mathrm{Obj}(\mathcal{C}(\mathcal{A}))$, is equivalent to the formula

$$\sum_{Q_\bullet} g^{Q_\bullet}_{M_\bullet, N_\bullet} g^{R_\bullet}_{Q_\bullet, P_\bullet} = \sum_{S_\bullet} g^{R_\bullet}_{M_\bullet, S_\bullet} g^{S_\bullet}_{N_\bullet, P_\bullet} \tag{1.3}$$

for any tuple $(M_\bullet, N_\bullet, P_\bullet, Q_\bullet)$ of objects of $\mathcal{C}(\mathcal{A})$.

We will sometimes use this renormalized generators $[[M_\bullet]]$ in the argument.

Hereafter we consider an abelian category $\mathcal{A}$ satisfying the conditions (a) – (c). Define $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P})) = (\mathcal{H}(\mathcal{C}(\mathcal{P})), *, [0])$ to be the pair of the vector space $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ with the twisted multiplication

$$[M_\bullet] * [N_\bullet] := t^{\langle M_0, N_0 \rangle + \langle M_1, N_1 \rangle} [M_\bullet] \diamond [N_\bullet] \tag{1.4}$$

and the class of the zero object. Here we used the Euler form $\langle A, B \rangle := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathfrak{k}} \mathrm{Ext}^i_{\mathcal{A}}(A, B)$ on $\mathcal{A}$. As is well known, this form descends to the one on the Grothendieck group $K(\mathcal{A})$ of $\mathcal{A}$, which is denoted by the same symbol. We will also use the symbol

$$\langle M_\bullet, N_\bullet \rangle' := \langle M_0, N_0 \rangle + \langle M_1, N_1 \rangle. \tag{1.5}$$

Now we can introduce Bridgeland's Hall algebra:

Definition/Fact 1.2. Let $\mathcal{DH}(\mathcal{A})$ be the localization of the algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$ with respect to the elements $[M_\bullet]$ corresponding to acyclic complexes $M_\bullet$:

$$\mathcal{DH}(\mathcal{A}) := \mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))\left[[M_\bullet]^{-1} \mid H_*(M_\bullet) = 0\right].$$

Then the tuple $(\mathcal{DH}(\mathcal{A}), [0], *)$ is a unital associative algebra.

Remark 1.3. Let us recall this localization process in detail. For $P \in \mathrm{Obj}(\mathcal{P})$, we define

$$K_{P_\bullet} := P \underset{0}{\overset{\mathrm{Id}}{\rightleftarrows}} P, \qquad K_{P_\bullet}^* := P \underset{-\mathrm{Id}}{\overset{0}{\rightleftarrows}} P,$$

which are obviously acyclic. By [Br1, Lemma 3.2], every acyclic complex of projectives $M_\bullet \in \mathrm{Obj}(\mathcal{C}(\mathcal{P}))$ can be written as

$$M_\bullet \cong K_{P_\bullet} \oplus K_{Q_\bullet}^*, \tag{1.6}$$

and $P, Q \in \mathrm{Obj}(\mathcal{P})$ are determined unique up to isomorphism. The complexes $K_{P_\bullet}$ and $K_{P_\bullet}^*$ enjoy the relations

$$[K_{P_\bullet}] * [M_\bullet] = t^{\langle \widehat{P}, \widehat{M}_\bullet \rangle} [K_{P_\bullet} \oplus M_\bullet] = t^{(\widehat{P}, \widehat{M}_\bullet)} M_\bullet * K_{P_\bullet},$$
$$[K_{P_\bullet}^*] * [M_\bullet] = t^{-\langle \widehat{P}, \widehat{M}_\bullet \rangle} [K_{P_\bullet}^* \oplus M_\bullet] = t^{-(\widehat{P}, \widehat{M}_\bullet)} M_\bullet * K_{P_\bullet}^*.$$

Here we used the symmetrized Euler form:

$$(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle. \tag{1.7}$$

Thus the subset $\{[M_\bullet] \mid H_*(M_\bullet) = 0\}$ of $\mathrm{Iso}(\mathcal{C}(\mathcal{P}))$ satisfies the Ore condition, and one can consider the localization of the non-commutative algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$ with respect to this subset. $\mathcal{DH}(\mathcal{A})$ is the algebra obtained by this localization process. As explained in [Br1, §3.6], this is the same as localizing by the elements $[K_{P_\bullet}]$ and $[K_{P_\bullet}^*]$ for all objects $P \in \mathrm{Obj}(\mathcal{P})$.

For an element $\alpha \in K(\mathcal{A})$, we define

$$K_\alpha := [K_{P_\bullet}] * [K_{Q_\bullet}]^{-1}, \quad K_\alpha^* := [K_{P_\bullet^*}] * [K_{Q_\bullet^*}]^{-1},$$

where we expressed $\alpha = \widehat{P} - \widehat{Q}$ using the classes of some projectives $P, Q \in \mathrm{Obj}(\mathcal{P})$. These are well defined by [Br1, Lemmas 3.4, 3.5], and also we have

$$K_\alpha * [M_\bullet] = t^{(\alpha, \widehat{M_\bullet})}[M_\bullet] * K_\alpha, \qquad K_\alpha^* * [M_\bullet] = t^{-(\alpha, \widehat{M_\bullet})}[M_\bullet] * K_\alpha, \tag{1.8}$$

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \qquad K_\alpha^* * K_\beta^* = K_{\alpha+\beta}^*, \tag{1.9}$$

$$[K_\alpha, K_\beta] = [K_\alpha, K_\beta^*] = [K_\alpha^*, K_\beta^*] = 0$$

in the algebra $\mathcal{DH}(\mathcal{A})$ for arbitrary $\alpha, \beta \in K(\mathcal{A})$ and $M_\bullet \in \mathrm{Obj}(\mathcal{C}(\mathcal{P}))$.

2. Coproduct

2.1. **Green's coproduct.** Let us recall the coalgebra structure on the ordinary Hall algebra introduced by Green [Gr]. Here one should consider a completion of the algebra. We recommend [Sc, Lecture 1] for a nice review on this topic.

Assume that the abelian category $\mathcal{A}$ satisfies the conditions (a), (b), (c) and (e). Then the Hall algebra $\mathcal{H}(\mathcal{A}) = \bigoplus_{A \in \mathrm{Iso}(\mathcal{A})} \mathbb{C}[A]$ is naturally graded by the Grothendieck group $K(\mathcal{A})$ of $\mathcal{A}$:

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{H}(\mathcal{A})[\alpha], \quad \mathcal{H}(\mathcal{A})[\alpha] := \bigoplus_{\widehat{A} = \alpha} \mathbb{C}[A].$$

For $\alpha, \beta \in K(\mathcal{A})$, set

$$\mathcal{H}(\mathcal{A})[\alpha] \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathcal{H}(\mathcal{A})[\beta] := \prod_{\widehat{A}=\alpha,\, \widehat{B}=\beta} \mathbb{C}[A] \otimes_{\mathbb{C}} \mathbb{C}[B], \qquad \mathcal{H}(\mathcal{A}) \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathcal{H}(\mathcal{A}) := \prod_{\alpha, \beta \in K(\mathcal{A})} \mathcal{H}(\mathcal{A})[\alpha] \mathbin{\widehat{\otimes}_{\mathbb{C}}} \mathcal{H}(\mathcal{A})[\beta].$$

Hereafter we will suppress the symbol $\mathbb{C}$ at the tensor product symbol $\otimes$. The space $\mathcal{H}(\mathcal{A}) \mathbin{\widehat{\otimes}} \mathcal{H}(\mathcal{A})$ consists of all formal linear combinations $\sum_{A,B} c_{A,B}[A] \otimes [B]$. The coassociativity in this completed space will be called the topological coassociativity.

Fact 2.1 (Green [Gr]). Assume that $\mathcal{A}$ satisfies the conditions (a), (b), (c) and (e).

(1) Then the maps

$$\Delta'_{\mathcal{H}(\mathcal{A})} \equiv \Delta' : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A}) \mathbin{\widehat{\otimes}} \mathcal{H}(\mathcal{A}),$$
$$\Delta'([A]) := \sum_{B,C} t^{\langle B,C \rangle} \frac{\left|\mathrm{Ext}^1_{\mathcal{A}}(B,C)_A\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(B,C)\right|} \frac{\left|\mathrm{Aut}_{\mathcal{A}}(A)\right|}{\left|\mathrm{Aut}_{\mathcal{A}}(B)\right|\left|\mathrm{Aut}_{\mathcal{A}}(C)\right|} [B] \otimes [C],$$

and

$$\epsilon : \mathcal{H}(\mathcal{A}) \longrightarrow \mathbb{C}, \qquad \epsilon([A]) := \delta_{A,0}.$$

define a topological counital coassociative coalgebra structure on $\mathcal{H}(\mathcal{A})$.

(2) Assume moreover that $\mathcal{A}$ also satisfies the condition (d). Then the tuple $(\mathcal{H}(\mathcal{A}), \diamond, [0], \Delta', \epsilon)$, is a topological bialgebra defined over $\mathbb{C}$. That is, the map $\Delta' : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{A}) \mathbin{\widehat{\otimes}} \mathcal{H}(\mathcal{A})$ is a homomorphism of $\mathbb{C}$-algebras.

Remark 2.2. As in Remark 1.1, we can rewrite the definition of our coproduct using $[[A]] := [A]/\left|\mathrm{Aut}_{\mathcal{A}}(A)\right|$ into the form

$$\Delta'([[A]]) = \sum_{B,C} t^{\langle B,C \rangle} g^A_{B,C} \frac{a_B a_C}{a_A} [[B]] \otimes [[C]] \tag{2.1}$$

with $a_A := \left|\mathrm{Aut}_{\mathcal{A}}(A)\right|$ and $g^A_{B,C} := \left|\{C' \subset A \mid C' \cong C,\ A/C' \cong B\}\right|$.

2.2. **Naive coproduct on Hall algebra of complexes.** To introduce a coproduct on Bridgeland's Hall algebra $\mathcal{DH}(\mathcal{A})$, we begin with the unlocalized algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$.

Definition 2.3. For an abelian category $\mathcal{A}$ satisfying the conditions (a), (b), (c) and (e), we define a $\mathbb{C}$-linear map

$$\Delta'_\chi : \mathcal{H}(\mathcal{C}(\mathcal{P})) \longrightarrow \mathcal{H}(\mathcal{C}(\mathcal{P})) \mathbin{\widehat{\otimes}} \mathcal{H}(\mathcal{C}(\mathcal{P}))$$

by

$$\Delta'_\chi([L_\bullet]) := \sum_{M_\bullet, N_\bullet} t^{\chi(M_\bullet, N_\bullet)} \frac{\left|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)_{L_\bullet}\right|}{\left|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)\right|} \frac{\left|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)\right|}{\left|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(M_\bullet)\right|\left|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(N_\bullet)\right|} [M_\bullet] \otimes [N_\bullet]. \tag{2.2}$$

Here χ is an arbitrary map from $\mathrm{Iso}(\mathcal{C}(\mathcal{P})) \times \mathrm{Iso}(\mathcal{C}(\mathcal{P}))$ to $\mathbb{Z}$ satisfying the condition

$$\begin{aligned} &\text{If} \quad \widehat{M_i} = \widehat{P_i} + \widehat{Q_i} \ \text{ and } \ \widehat{R_i} = \widehat{Q_i} + \widehat{N_i} \ \text{ for } \ i = 0, 1, \\ &\text{then} \quad \chi(M_\bullet, N_\bullet) + \chi(P_\bullet, Q_\bullet) = \chi(P_\bullet, R_\bullet) + \chi(Q_\bullet, N_\bullet). \end{aligned} \tag{2.3}$$

We also define

$$\epsilon : \mathcal{H}(\mathcal{C}(\mathcal{P})) \longrightarrow \mathbb{C}$$

by

$$\epsilon([M_\bullet]) := \delta_{M_\bullet, 0}. \tag{2.4}$$

Remark 2.4. (1) Examples for χ is the Euler form $\langle \cdot, \cdot \rangle'$ appearing in (1.5). We can transpose the entries and can multiply the form by scalar. So $\chi(M_\bullet, N_\bullet) := -\langle N_\bullet, M_\bullet \rangle'$ also satisfies the condition (2.3).

(2) As in Remark 1.1, we can rewrite our coproduct using $[[L_\bullet]] := [L_\bullet]/\big|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)\big|$ into the form

$$\Delta'_\chi([[L_\bullet]]) = \sum_{M_\bullet, N_\bullet} t^{\chi(M_\bullet, N_\bullet)} g^{L_\bullet}_{M_\bullet, N_\bullet} \frac{a_{M_\bullet} a_{N_\bullet}}{a_{L_\bullet}} [[M_\bullet]] \otimes [[N_\bullet]]$$

with

$$a_{L_\bullet} := \big|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)\big|. \tag{2.5}$$

We will often use the symbols $[[L_\bullet]]$ and $a_{L_\bullet}$ in the following argument.

Lemma 2.5. *Assume that an abelian category $\mathcal{A}$ satisfying the conditions (a), (b), (c) and (e). Then the triple $\big(\mathcal{H}(\mathcal{C}(\mathcal{P})), \Delta'_\chi, \epsilon\big)$ is a topological counital coassociative coalgebra.*

Proof. Our argument is almost the same as that for the proof of the coassociativity of the ordinary Hall algebra (Fact 2.1(1)), but for completeness we will write down a proof.

The topological coassociativity of Δ'_χ, that is $(\Delta'_\chi \otimes 1) \circ \Delta'_\chi([[L_\bullet]]) = (1 \otimes \Delta'_\chi) \circ \Delta'_\chi([[L_\bullet]])$ on the completed tensor product $\mathcal{H}(\mathcal{C}(\mathcal{P})) \,\widehat{\otimes}\, \mathcal{H}(\mathcal{C}(\mathcal{P})) \,\widehat{\otimes}\, \mathcal{H}(\mathcal{C}(\mathcal{P}))$ for any object $L_\bullet$ in $\mathcal{H}(\mathcal{C}(\mathcal{P}))$, is equivalent to the following formula

$$\begin{aligned} &\sum_{M_\bullet, N_\bullet, P_\bullet, Q_\bullet} t^{\chi(M_\bullet, N_\bullet)} g^{L_\bullet}_{M_\bullet, N_\bullet} \frac{a_{M_\bullet} a_{N_\bullet}}{a_{L_\bullet}} t^{\chi(P_\bullet, Q_\bullet)} g^{M_\bullet}_{P_\bullet, Q_\bullet} \frac{a_{P_\bullet} a_{Q_\bullet}}{a_{M_\bullet}} [[P_\bullet]] \otimes [[Q_\bullet]] \otimes [[N_\bullet]] \\ = &\sum_{M_\bullet, N_\bullet, P_\bullet, Q_\bullet} t^{\chi(M_\bullet, N_\bullet)} g^{L_\bullet}_{M_\bullet, N_\bullet} \frac{a_{M_\bullet} a_{N_\bullet}}{a_{L_\bullet}} t^{\chi(P_\bullet, Q_\bullet)} g^{N_\bullet}_{P_\bullet, Q_\bullet} \frac{a_{P_\bullet} a_{Q_\bullet}}{a_{N_\bullet}} [[M_\bullet]] \otimes [[P_\bullet]] \otimes [[Q_\bullet]]. \end{aligned}$$

One can see that it is equivalent to the formula

$$\sum_{M_\bullet} t^{\chi(M_\bullet, N_\bullet) + \chi(P_\bullet, Q_\bullet)} g^{L_\bullet}_{M_\bullet, N_\bullet} g^{M_\bullet}_{P_\bullet, Q_\bullet} \frac{a_{N_\bullet} a_{P_\bullet} a_{Q_\bullet}}{a_{L_\bullet}} = \sum_{R_\bullet} t^{\chi(P_\bullet, R_\bullet) + \chi(Q_\bullet, N_\bullet)} g^{L_\bullet}_{P_\bullet, R_\bullet} g^{R_\bullet}_{Q_\bullet, N_\bullet} \frac{a_{N_\bullet} a_{P_\bullet} a_{Q_\bullet}}{a_{L_\bullet}}$$

for any tuple $(L_\bullet, N_\bullet, P_\bullet, Q_\bullet)$ of objects in $\mathcal{C}(\mathcal{P})$. By the condition (2.3), the last formula is reduced to

$$\sum_{M_\bullet} g^{M_\bullet}_{P_\bullet, Q_\bullet} g^{L_\bullet}_{M_\bullet, N_\bullet} = \sum_{R_\bullet} g^{L_\bullet}_{P_\bullet, R_\bullet} g^{R_\bullet}_{Q_\bullet, N_\bullet},$$

which is nothing but the consequence of the associativity (1.3) of the product $\diamond$.

It is easy to see that ϵ gives a counit. Thus we have the conclusion. □

Remark 2.6. Now we want to consider the localized algebra $\mathcal{DH}(\mathcal{A}) = \mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))\big[[M_\bullet]^{-1} \mid H_*(M_\bullet) = 0\big]$. Let us denote by

$$S := \{[M_\bullet] \mid H_*(M_\bullet) = 0\}$$

the subset of $C := \mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$ used in the localization. If S spans a coideal with respect to the coproduct Δ'_χ, in other words $\Delta'_\chi(S) \subset S \,\widehat{\otimes}\, C + C \,\widehat{\otimes}\, S$, then the coalgebra structure $\big(\mathcal{H}(\mathcal{C}(\mathcal{P})), \Delta'_\chi\big)$ descends to $\big(C/\operatorname{Span}(S), \Delta'_\chi\big)$.

However, this strategy does not work. Consider the exact sequence

$$0 \longrightarrow N_\bullet \longrightarrow L_\bullet \longrightarrow M_\bullet \longrightarrow 0$$

in $\mathcal{C}(\mathcal{P})$, which is expressed as the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d, shift left] & P \arrow[r, "\mathrm{Id}_P"] \arrow[d, shift left, "0"] & P \arrow[r] \arrow[d, shift left] & 0 \\ 0 \arrow[r] & P \arrow[r, "\mathrm{Id}_P"'] \arrow[u, shift left] & P \arrow[r] \arrow[u, shift left, "\mathrm{Id}_P"] & 0 \arrow[r] \arrow[u, shift left] & 0 \end{tikzcd} \tag{2.6}$$

with $P \in \mathrm{Obj}(\mathcal{P})$. In this case we have $H_*(L_\bullet) = 0$ but $H_*(M_\bullet) \neq 0$ and $H_*(N_\bullet) \neq 0$. Thus we have $\Delta'_\chi(S) \not\subset S \,\widehat{\otimes}\, C + C \,\widehat{\otimes}\, S$.

In the next subsection we impose certain conditions on the exact sequences in $\mathcal{C}(\mathcal{P})$ which are counted in the desired coproduct formula. For such a purpose, it is convenient to recall the notion of exact categories in the sense of Quillen [Qu].

2.3. **Hall algebras of exact categories and descent of coproduct.** We follow the description of an exact category given in [Hu, §2] and [Ke, Appendix A].

An exact category in the sense of Quillen [Qu] is a pair $(\mathcal{A}, \mathcal{E})$ of an additive category $\mathcal{A}$ and a class $\mathcal{E}$ of kernel-cokernel pairs (f, g) closed under isomorphisms, satisfying the following axioms. A deflection mentioned in these axioms is the first component of some $(f, g) \in \mathcal{E}$, and a inflation is the second component. A pair $(f, g) \in \mathcal{E}$ will be called a conflation.

(Ex0) $0 \xrightarrow{\mathrm{Id}} 0$ is a deflation.
(Ex1) The composition of two deflations is a deflation.
(Ex2) For any $f : Z' \to Z$ and a deflation $d : Y \to Z$, there is a cartesian square

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"'] & Z \end{tikzcd}$$

with d' a deflation.
(Ex2^{op}) For any $f : X \to X'$ and each inflation $i : X \to Y$, there is a cocartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

with i' an inflation.

The notion of Hall algebra for an exact category was introduced in [Hu], and the associativity of the algebra was shown:

Fact 2.7 ([Hu, Theorem. 3])**.** Let $(\mathcal{A}, \mathcal{E})$ be an exact category with $\mathcal{A}$ essentially small and having finite morphism spaces.

Let us define a free $\mathbb{Z}$-module

$$\mathcal{H}(\mathcal{A}, \mathcal{E}) := \bigoplus_{X \in \mathrm{Iso}(\mathcal{A})} \mathbb{Z}[X]$$

and introduce a binary operator $\diamond$ by

$$[X] \diamond [Y] := \sum_{Z} \frac{\left|W_{X,Y}^{Z}\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(X, Y)\right|}[Z].$$

Here $W_{X,Y}^{Z}$ is the set of all conflations of the form $Y \to Z \to X$.

Then $(\mathcal{H}(\mathcal{A}, \mathcal{E}), \diamond, [0])$ is a unital associative ring.

Of course the Hall algebra $\mathcal{H}(\mathcal{A})$ for an abelian category $\mathcal{A}$ satisfying the conditions (a) coincides with $\mathcal{H}(\mathcal{A}, \mathcal{E})$ (after tensoring $\mathbb{C}$), where $\mathcal{E}$ is the set of all exact sequences in the abelian category $\mathcal{A}$.

Now we want to introduce a special subset $\mathcal{E}_0$ of the set of all exact sequences in $\mathcal{C}(\mathcal{P})$, and to define the Hall algebra associated to exact category $\big(\mathcal{C}(\mathcal{P}), \mathcal{E}\big)$

Definition 2.8. Let $\mathcal{E}_0$ be the class of exact sequences

$$0 \longrightarrow N_\bullet \longrightarrow L_\bullet \longrightarrow M_\bullet \longrightarrow 0$$

in $\mathcal{C}(\mathcal{P})$ satisfying the condition

$$H_i(M) \neq 0 \Longrightarrow H_{i+1}(N) = 0 \;\text{ for both }\; i = 0, 1. \tag{2.7}$$

The motivation of this definition comes from Remark 2.6. The exact sequence (2.6) is excluded from $\mathcal{E}_0$ by the condition (2.7).

Proposition 2.9. *The pair $\big(\mathcal{C}(\mathcal{P}), \mathcal{E}_0\big)$ is an exact category.*

Proof. The axioms (Ex0) and (Ex1) are trivially true. $\mathcal{C}(\mathcal{P})$ is closed under cartesian and cocartesian squares, so it is enough to check that for a diagram

$$\begin{tikzcd} 0 \arrow[r] & N'_\bullet \arrow[r, "i'"] \arrow[d] \arrow[dr, phantom, "\square"] & L'_\bullet \arrow[r, "d'"] \arrow[d] \arrow[dr, phantom, "\square"] & M'_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N_\bullet \arrow[r, "i"'] & L_\bullet \arrow[r, "d"'] & M_\bullet \arrow[r] & 0 \end{tikzcd}$$

if the lower row belongs to $\mathcal{E}_0$ then the upper row belongs to $\mathcal{E}_0$ and vice-a-versa. But it can be checked by simple diagram chasing. □

Now we introduce

Definition 2.10. For an abelian category $\mathcal{A}$ satisfying the conditions (a) – (c), denote by $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P},\mathcal{E}_0)) = (\mathcal{H}(\mathcal{C}(\mathcal{P})), *_{\mathcal{E}_0})$ the $\mathbb{C}$-vector space $\mathcal{H}(\mathcal{C}(\mathcal{P})) = \bigoplus_{[M_\bullet]\in \mathrm{Iso}(\mathcal{C}(\mathcal{P}))} \mathbb{C}[M_\bullet]$ with the multiplication

$$[M_\bullet] *_{\mathcal{E}_0} [N_\bullet] := t^{\langle M_\bullet, N_\bullet\rangle'} \sum_{L_\bullet \in \mathrm{Iso}(\mathcal{C}(\mathcal{P}))} \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)|}[L_\bullet].$$

Here $W^{L_\bullet}_{M_\bullet,N_\bullet}$ denotes the set of all conflations $N_\bullet \to L_\bullet \to M_\bullet$ in $\mathcal{E}_0$.

By Fact 2.7, $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P},\mathcal{E}_0))$ is a unital associative algebra.

Using the exact category $\big(\mathcal{C}(\mathcal{P}), \mathcal{E}_0\big)$, we introduce a coproduct on the Hall algera of complexes.

Definition 2.11. For an abelian category $\mathcal{A}$ satisfying the conditions (a), (b), (c) and (e), we define a $\mathbb{C}$-linear map

$$\Delta'_{\chi,\mathcal{E}_0} : \mathcal{H}(\mathcal{C}(\mathcal{P})) \longrightarrow \mathcal{H}(\mathcal{C}(\mathcal{P}))\,\widehat{\otimes}\,\mathcal{H}(\mathcal{C}(\mathcal{P}))$$

by

$$\Delta'_{\chi,\mathcal{E}_0}([L_\bullet]) := \sum_{M_\bullet,N_\bullet} t^{\chi(M_\bullet,N_\bullet)} \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)|} \frac{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)|}{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(M_\bullet)||\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(N_\bullet)|}[M_\bullet]\otimes[N_\bullet].$$

where $W^{L_\bullet}_{M_\bullet,N_\bullet}$ denotes the set of all conflations $N_\bullet \to L_\bullet \to M_\bullet$ in $\mathcal{E}_0$, and χ is an arbitrary map from $\mathrm{Iso}(\mathcal{C}(\mathcal{P})) \times \mathrm{Iso}(\mathcal{C}(\mathcal{P}))$ to $\mathbb{Z}$ satisfying the condition (2.3).

Remark 2.12. As in Remark 1.1, we can rewrite the coproduct in the next equivalent form:

$$\Delta'_{\chi,\mathcal{E}_0}([[L_\bullet]]) = \sum_{M_\bullet,N_\bullet} t^{\chi(M_\bullet,N_\bullet)} w^{L_\bullet}_{M_\bullet,N_\bullet} \frac{a_{M_\bullet} a_{N_\bullet}}{a_{L_\bullet}}[M_\bullet]\otimes[N_\bullet]$$

with

$$w^{L_\bullet}_{M_\bullet,N_\bullet} := \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet, N_\bullet)|} \frac{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(M_\bullet)||\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(N_\bullet)|}{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)|}. \tag{2.8}$$

Note that $w^{L_\bullet}_{M_\bullet,N_\bullet}$ is a substitute of $g^{L_\bullet}_{M_\bullet,N_\bullet}$ (1.1).

As in Lemma 2.5, we have

Lemma 2.13. *Assume that an abelian category $\mathcal{A}$ satisfies the conditions (a), (b), (c) and (e). Then $\big(\mathcal{H}(\mathcal{C}(\mathcal{P})), \Delta'_{\chi,\mathcal{E}_0}\big)$ is a topological coassociative coalgebra over $\mathbb{C}$.*

Recalling the proof of Lemma 2.5 we note that this lemma is a consequence of the associativity of the algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}) = \big(\mathcal{H}(\mathcal{C}(\mathcal{P})), *_{\mathcal{E}_0}\big)$.

Now we want to descend the coproduct $\Delta'_{\chi,\mathcal{E}_0}$ of $\mathcal{H}(\mathcal{C}(\mathcal{P}))$. to the localized algebra $\mathcal{DH}(\mathcal{A})$. We have

Proposition 2.14. *For an abelian category $\mathcal{A}$ satisfying the conditions (a), (b), (c) and (e), the subset*

$$S := \{[M_\bullet] \mid H_*(M_\bullet) = 0\}$$

of $\mathcal{H}(\mathcal{C}(\mathcal{P}))$ spans a coideal of the coalgebra $\big(\mathcal{H}(\mathcal{C}(\mathcal{P})), \Delta'_{\chi,\mathcal{E}_0}\big)$,

Proof. By (1.6), we can express any element in S as $[K_{P_\bullet} \oplus K^*_{Q_\bullet}]$ with some $P, Q \in \mathrm{Obj}(\mathcal{P})$. So we study the short exact sequence

$$0 \longrightarrow N_\bullet \longrightarrow K_{P_\bullet} \oplus K^*_{Q_\bullet} \longrightarrow M_\bullet \longrightarrow 0 \tag{2.9}$$

in $\mathcal{C}(\mathcal{P})$. What should be shown is that $[M_\bullet]\otimes[N_\bullet] \in S\otimes C + C\otimes S$ if (2.9) appears in $\mathcal{E}_0$.

The long exact sequence in homology induced from (2.9) can be split to give two long exact sequences

$$\begin{array}{c} 0 \longrightarrow K \longrightarrow H_1(N_\bullet) \longrightarrow 0 \longrightarrow H_1(M_\bullet) \longrightarrow C \longrightarrow 0\,. \\ 0 \longrightarrow C \longrightarrow H_0(N_\bullet) \longrightarrow 0 \longrightarrow H_0(M_\bullet) \longrightarrow K \longrightarrow 0 \end{array}$$

Here we used the 2-periodicity of complexes. Thus we have $H_1(M_\bullet) = H_0(N_\bullet)$ and $H_0(M_\bullet) = H_1(N_\bullet)$. Recalling the condition (2.7), we conclude that no exact sequence of the form (2.9) appears in $\mathcal{E}_0$. $\square$

2.4. **Genuine coproduct.** Using the formulation of quotient coalgebra presented in the last subsection, we introduce a good coproduct on the whole algebra $\mathcal{DH}(\mathcal{A})$.

As a preliminary, we have

Lemma 2.15. *For an abelian category $\mathcal{A}$ satisfying the conditions (a), any object $M_\bullet$ in $\mathcal{C}(\mathcal{P})$ is of the form $M_\bullet \cong N_\bullet \oplus K_\bullet$, where $N_\bullet$ and $K_\bullet$ are objects in $\mathcal{C}(\mathcal{P})$ and $K_\bullet$ is a maximal acyclic subobject of $M_\bullet$.*

Proof. Since the condition (a) ensures that $\mathcal{A}$ enjoys Krull-Schmidt property, $\mathcal{C}(\mathcal{A})$ is also a Krull-Schmidt category. Then the assertion is trivial. $\square$

By [Br1, Lema 3.2] every acyclic object $K_\bullet$ in $\mathcal{C}(\mathcal{P})$ can be expressed as $K_{P_\bullet} \oplus K_{Q_\bullet}^{\ *}$. Recall also that for acyclic $K_\bullet$ and any $N_\bullet$ we have $\mathrm{Ext}^1_{\mathcal{C}(\mathcal{P})}(N_\bullet, K_\bullet) = \mathrm{Ext}^1_{\mathcal{C}(\mathcal{P})}(K_\bullet, N_\bullet) = 0$. Combining these results we have

Lemma 2.16. *For an abelian category $\mathcal{A}$ satisfying the conditions (a) – (c), $\mathcal{DH}(\mathcal{A})$ has a basis consisting of elements*

$$[N_\bullet] * K_\alpha * K_\beta^*$$

with $H^*(N_\bullet) \neq 0$ and $\alpha, \beta \in K(\mathcal{A})$.

Now we introduce a coproduct on the whole algebra $\mathcal{DH}(\mathcal{A})$.

Definition 2.17. Let $\mathcal{A}$ be an abelian category satisfying the conditions (a), (b), (c) and (e), and let χ be an arbitrary map from $\mathrm{Iso}(\mathcal{C}(\mathcal{P})) \times \mathrm{Iso}(\mathcal{C}(\mathcal{P}))$ to $\mathbb{Z}$ (or $\mathbb{C}$) satisfying the condition (2.3).

Define a $\mathbb{C}$-linear map

$$\Delta'_\chi : \mathcal{DH}(\mathcal{A}) \longrightarrow \mathcal{DH}(\mathcal{A}) \,\widehat{\otimes}\, \mathcal{DH}(\mathcal{A})$$

by

$$\Delta'_\chi([N_\bullet] * K_\alpha * K_\beta^*) := \Delta'_{\chi,\mathcal{E}^0}([N_\bullet]) * (K_\alpha \otimes K_\alpha) * (K_\beta^* \otimes K_\beta^*).$$

Here we used Lemma 2.16 and the multiplication

$$(x \otimes y) * (z \otimes w) = (x * z) \otimes (y * w)$$

on the tensor space $\mathcal{DH}(\mathcal{A}) \,\widehat{\otimes}\, \mathcal{DH}(\mathcal{A})$.

We also define a $\mathbb{C}$-linear map

$$\epsilon : \mathcal{DH}(\mathcal{A}) \longrightarrow \mathbb{C}$$

by

$$\epsilon([N_\bullet] * K_\alpha * K_\beta^*) := \delta_{N_\bullet,[0]}.$$

Theorem 2.18. *Assume that $\mathcal{A}$ is an abelian category satisfying the conditions (a) – (e). Then the tuple $(\mathcal{DH}(\mathcal{A}), \Delta'_\chi, \epsilon)$ is a topological counital coassociative coalgebra,*

Proof. Let us denote $S := \{[M_\bullet] \mid H_*(M_\bullet) = 0\}$. Proposition 2.14 yields that $\Delta'_{\chi,\mathcal{E}_0}$ descends to the quotient $\mathbb{C}$-vector space $\mathcal{H}(\mathcal{C}(\mathcal{P}))/\operatorname{Span}(S)$. By Lemma 2.16, this quotient space has the basis consisting of $[N_\bullet]$. With this observation and the Ore condition satisfied by S, we see that Δ'_χ is well-defined and $(\mathcal{DH}(\mathcal{A}), \Delta'_\chi)$ is a topological coassociative coalgebra.

It is easy to check that ϵ gives a counit. $\square$

3. Hereditary case

In the case where $\mathcal{A}$ is hereditary, one knows that $\mathcal{H}(\mathcal{A})$ is embedded into $\mathcal{DH}(\mathcal{A})$ as an algebra [Br1, Lemma 4.3], and moreover $\mathcal{DH}(\mathcal{A})$ is Drinfeld double of $\mathcal{H}(\mathcal{A})$ [Ya]. In this section we compare the coalgebra structures on $\mathcal{H}(\mathcal{A})$ and $\mathcal{DH}(\mathcal{A})$.

3.1. **Basis of $\mathcal{DH}(\mathcal{A})$.** Assume that $\mathcal{A}$ satisfies the conditions (a)–(e). By [Br1, §4] we have a nice basis for $\mathcal{DH}(\mathcal{A})$. To explain that, let us recall the minimal resolution of objects of $\mathcal{A}$.

Definition 3.1 ([Br1, §4])**.** Assume the conditions (a), (c) and (d) on $\mathcal{A}$.

(1) Every object $A \in \mathcal{A}$ has a projective resolution

$$0 \longrightarrow P \xrightarrow{f} Q \longrightarrow A \longrightarrow 0. \tag{3.1}$$

Decomposing P and Q into finite direct sums $P = \oplus_i P_i$, $Q = \oplus_j Q_j$, one may write $f = (f_{ij})$ in matrix form with $f_{ij} : P_i \to Q_j$. The resolution (3.1) is said to be minimal if none of the morphisms f_{ij} is an isomorphism.

(2) Given an object A in $\mathcal{A}$, take a minimal projective resolution

$$0 \longrightarrow P_A \xrightarrow{f_A} Q_A \longrightarrow A \longrightarrow 0,$$

we define a $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$C_{A_\bullet} := \; P_A \underset{0}{\overset{f_A}{\rightleftarrows}} Q_A \;\in \mathrm{Obj}(\mathcal{C}(\mathcal{P})). \tag{3.2}$$

(Remark: By [Br1, Lemma 4.1], arbitrary two minimal projective resolutions of A are isomorphic, so the complex $C_{A_\bullet}$ is well-defined up to isomorphism.)

(3) Assume $\mathcal{A}$ satisfies the conditions (a) – (e). Given an object A in $\mathcal{A}$, we define elements $E_{A_\bullet}, F_{A_\bullet}$ in $\mathcal{DH}(\mathcal{A})$ by

$$E_A := t^{\langle P_A, A\rangle} K_{-\widehat{P_A}} * [C_{A_\bullet}], \qquad F_A := {E_A}^*. \tag{3.3}$$

Here we used a minimal projective decomposition of A and the associated complex $C_{A_\bullet}$ shown in (3.2).

Fact 3.2 ([Br1, Lemma 4.2])**.** Assume $\mathcal{A}$ is an abelian category satisfying the conditions (a), (c) and (d). Then every object $M_\bullet$ in $C(\mathcal{P})$ has a direct sum decomposition

$$M_\bullet = C_{A_\bullet} \oplus C_{B_\bullet}^* \oplus K_{P_\bullet} \oplus K_{Q_\bullet}^*.$$

Moreover, the objects $A, B \in \mathrm{Obj}(\mathcal{A})$ and $P, Q \in \mathrm{Obj}(\mathcal{P})$ are unique up to isomorphism.

One also has

Fact 3.3 (Corollary of [Br1, Lemmas 4.6, 4.7])**.** $\mathcal{DH}(\mathcal{A})$ has a basis consisting of elements

$$E_A * K_\alpha * K_\beta^* * F_B, \quad A, B \in \mathrm{Iso}(\mathcal{A}),\ \alpha, \beta \in K(\mathcal{A}).$$

3.2. **Twisted coproduct and coalgebra embedding.** Let us recall the twisted Hall algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{A})$ for an abelian category $\mathcal{A}$.

Definition 3.4. Let $\mathcal{A}$ be an abelian category satisfying the conditions (a) –(c). The twisted Hall algebra $\mathcal{H}_{\mathrm{tw}}(\mathcal{A}) = \big(\mathcal{H}(\mathcal{A}), *, [0]\big)$ is the tuple consisting of the $\mathbb{C}$-vector space $\mathcal{H}(\mathcal{A})$, the twisted multiplication

$$[A] * [B] := t^{\langle \widehat{A}, \widehat{B}\rangle} [A] \diamond [B] \tag{3.4}$$

for $A, B \in \mathrm{Iso}(\mathcal{A})$, and the class of the zero object.

Remark 3.5. As mentioned in [Br1, §3.5], the symbol $*$ has different meanings in $\mathcal{H}_{\mathrm{tw}}(\mathcal{A})$ and $\mathcal{H}_{\mathrm{tw}}(\mathcal{C}(\mathcal{P}))$. Compare the expressions (1.4) and (3.4).

As for the relation to Bridgeland's Hall algebra, we have

Fact 3.6 ([Br1, Lemma 4.3])**.** Assume that $\mathcal{A}$ satisfies the conditions (a) – (e). Then there is an embedding of $\mathbb{C}$-algebras

$$I_+^{\mathrm{e}} : \big(\mathcal{H}_{\mathrm{tw}}(\mathcal{A}), *, [0]\big) \longhookrightarrow \big(\mathcal{DH}(\mathcal{A}), *, [0]\big) \qquad [A] \longmapsto E_A. \tag{3.5}$$

Next let us recall the extended Hall algebra $\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A})$ and the twisted coproduct Δ on it (see also [Sc, §1.5, Page 16] for the explanation).

The extended Hall algebra $\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A})$ is defined as an extension of $\mathcal{H}_{\mathrm{tw}}(\mathcal{A})$ by adjoining symbols K_α for classes $\alpha \in K(\mathcal{A})$, and imposing relations

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \qquad K_\alpha * [B] = t^{(\alpha, \widehat{B})} [B] * K_\alpha$$

for $\alpha, \beta \in K(\mathcal{A})$ and $B \in \mathrm{Iso}(\mathcal{A})$. Here we used the symmetrized Euler form (1.7). Thus $\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A})$ has a vector space basis consisting of the elements $K_\alpha * [B]$ for $\alpha \in K(\mathcal{A})$ and $B \in \mathrm{Iso}(\mathcal{A})$.

Fact 3.7. Assume that $\mathcal{A}$ satisfies the conditions (a), (b), (c) and (e).

(1) Then the maps

$$\Delta_{\mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A})} \equiv \Delta : \mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}) \longrightarrow \mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}) \widehat{\otimes} \mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}),$$

$$\Delta([A] * K_\alpha) := \sum_{B,C} t^{\langle \widehat{B}, \widehat{C} \rangle} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(B,C)_A|}{|\mathrm{Hom}_{\mathcal{A}}(B,C)|} \frac{a_A}{a_B a_C} \big([B] * K_{\widehat{C}+\alpha}\big) \otimes \big([C] * K_\alpha\big), \tag{3.6}$$

and

$$\epsilon : \mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}) \longrightarrow \mathbb{C}, \qquad \epsilon([A] * K_\alpha) := \delta_{A,0},$$

define a topological counital coassociative coalgebra structure on $\mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A})$.

(2) Assume moreover that $\mathcal{A}$ also satisfies the condition (d). Then the tuple $(\mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}), *, [0], \Delta, \epsilon)$, is a topological bialgebra defined over $\mathbb{C}$.

The relation to Bridgeland's Hall algebra is described by

Fact 3.8 ([Br1, Lemma 4.6])**.** Assume that $\mathcal{A}$ satisfies the conditions (a) – (e). Then there is an embedding of algebras

$$I^{\rm e}_+ : \big(\mathcal{H}^{\rm e}_{\rm tw}(\mathcal{A}), *, [0]\big) \hookrightarrow \big(\mathcal{DH}(\mathcal{A}), *, [0]\big) \tag{3.7}$$

defined on generators by $[A] \mapsto E_A$ and $K_\alpha \mapsto K_\alpha$.

Now we introduce a twisted coproduct on $\mathcal{DH}(\mathcal{A})$ using the basis shown in Fact 3.2 and Fact 3.3.

Definition 3.9. Let $\mathcal{A}$ be an abelian category satisfying the conditions (a) – (e), and let χ be an arbitrary map from $\mathrm{Iso}(\mathcal{C}(\mathcal{P})) \times \mathrm{Iso}(\mathcal{C}(\mathcal{P}))$ to $\mathbb{Z}$ (or $\mathbb{C}$) satisfying the condition (2.3).

Define a $\mathbb{C}$-linear map

$$\Delta_\chi : \mathcal{DH}(\mathcal{A}) \longrightarrow \mathcal{DH}(\mathcal{A}) \widehat{\otimes} \mathcal{DH}(\mathcal{A})$$

by

$$\Delta_\chi([C_{A_\bullet} \oplus C_{B_\bullet}^*] * K_\alpha * K_\beta^*) := \Delta_{\chi,\mathcal{E}^0}([C_{A_\bullet}])\Delta^{op}_{\chi,\mathcal{E}^0}([C_{B_\bullet}]) * (K_\alpha \otimes K_\alpha) * (K_\beta^* \otimes K_\beta^*).$$

with

$$\Delta_{\chi,\mathcal{E}^0}([L_\bullet]) := \sum_{M_\bullet, N_\bullet} t^{\chi(M_\bullet,N_\bullet)} \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet,N_\bullet)|} \frac{a_{L_\bullet}}{a_{M_\bullet} a_{N_\bullet}} \big(K_{\widehat{N_0}} * [M_\bullet]\big) \otimes \big([N_\bullet] * K_{\widehat{M_1}}\big),$$

$$\Delta^{op}_{\chi,\mathcal{E}^0}([L_\bullet]) := \sum_{M_\bullet, N_\bullet} t^{\chi(M_\bullet,N_\bullet)} \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet,N_\bullet)|} \frac{a_{L_\bullet}}{a_{M_\bullet} a_{N_\bullet}} \big(K^*_{\widehat{N_1}} * [M_\bullet]\big) \otimes \big([N_\bullet] * K^*_{\widehat{M_0}}\big),$$

where $W^{L_\bullet}_{M_\bullet,N_\bullet}$ denotes the set of all conflations $N_\bullet \to L_\bullet \to M_\bullet$ in $\mathcal{E}_0$, and we used the multiplication

$$(x \otimes y) * (z \otimes w) = (x * z) \otimes (y * w)$$

on the tensor space $\mathcal{DH}(\mathcal{A}) \widehat{\otimes} \mathcal{DH}(\mathcal{A})$.

Then we have

Proposition 3.10. *For an abelian category $\mathcal{A}$ satisfying the conditions (a), (b), (c) and (e), the tuple $(\mathcal{DH}(\mathcal{A}), \Delta_\chi, \epsilon)$ is a topological coassociative coalgebra.*

Proof. Let us show $(\Delta \otimes 1) \circ \Delta([[L_\bullet]]) = (1 \otimes \Delta) \circ \Delta([[L_\bullet]])$ for $L_\bullet = C_{A_\bullet}$. The other cases are similar. As in the proof of Lemma 2.5, it is equivalent to the formula

$$\begin{aligned}
&\sum_{M_\bullet} t^{\chi(M_\bullet,N_\bullet)+\chi(P_\bullet,Q_\bullet)} w^{L_\bullet}_{M_\bullet,N_\bullet} w^{M_\bullet}_{P_\bullet,Q_\bullet} \frac{a_{N_\bullet} a_{P_\bullet} a_{Q_\bullet}}{a_{L_\bullet}} \\
&\qquad \big(K_{\widehat{N_0}} * K_{\widehat{Q_0}} * [[P_\bullet]]\big) \otimes \big(K_{\widehat{N_0}} * [[Q_\bullet]] * K_{\widehat{P_1}}\big) \otimes \big([[N_\bullet]] * K_{\widehat{M_1}}\big) \\
=&\sum_{R_\bullet} t^{\chi(P_\bullet,R_\bullet)+\chi(Q_\bullet,N_\bullet)} w^{L_\bullet}_{P_\bullet,R_\bullet} w^{R_\bullet}_{Q_\bullet,N_\bullet} \frac{a_{N_\bullet} a_{P_\bullet} a_{Q_\bullet}}{a_{L_\bullet}} \\
&\qquad \big(K_{\widehat{R_0}} * [[P_\bullet]]\big) \otimes \big(K_{\widehat{N_0}} * [[Q_\bullet]] * K_{\widehat{P_1}}\big) \otimes \big([[N_\bullet]] * K_{\widehat{Q_1}} * K_{\widehat{P_1}}\big)
\end{aligned}$$

with

$$w^{L_\bullet}_{M_\bullet,N_\bullet} := \frac{|W^{L_\bullet}_{M_\bullet,N_\bullet}|}{|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet,N_\bullet)|} \frac{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(M_\bullet)||\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(N_\bullet)|}{|\mathrm{Aut}_{\mathcal{C}(\mathcal{A})}(L_\bullet)|}$$

for any tuple $(L_\bullet, N_\bullet, P_\bullet, Q_\bullet)$ of objects in $\mathcal{C}(\mathcal{P})$ satisfying the condition $\widehat{M_i} = \widehat{P_i} + \widehat{Q_i}$ and $\widehat{R_i} = \widehat{Q_i} + \widehat{N_i}$ for $i = 0, 1$. This condition tells us that the coassociativity follows from the formula $\sum_{M_\bullet} w^{L_\bullet}_{M_\bullet,N_\bullet} w^{M_\bullet}_{P_\bullet,Q_\bullet} = \sum_{R_\bullet} w^{L_\bullet}_{P_\bullet,R_\bullet} w^{R_\bullet}_{Q_\bullet,N_\bullet}$, which is the associativity of $\mathcal{H}(\mathcal{C}(\mathcal{P}), \mathcal{E}_0)$. ∎

Our claim is

Theorem 3.11. *For an abelian category $\mathcal{A}$ satisfying the conditions (a) – (e), the map I_+^{e} (3.7) defines an embedding of $\mathbb{C}$-coalgebras*

$$I_+^{\mathrm{e}} : \big(\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A}), \Delta_{\mathcal{H}(\mathcal{A})}, \epsilon\big) \longrightarrow \big(\mathcal{DH}(\mathcal{A}), \Delta_{\chi_0}, \epsilon\big).$$

Here we use

$$I_+^{\mathrm{e}}(x \otimes y) := I_+^{\mathrm{e}}(x) \otimes I_+^{\mathrm{e}}(y)$$

on the tensor product space $\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A}) \mathbin{\widehat{\otimes}} \mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A})$ and

$$\chi_0(M_\bullet, N_\bullet) := -\langle N_\bullet, M_\bullet \rangle.$$

Remark 3.12. (1) As mentioned in Remark 2.4 (1), $\chi = \chi_0$ satisfies the condition (2.3) imposed on the map χ.
(2) As in Remark 2.4, one can rewrite the coproduct Δ_{χ_0} using the notation $[[L_\bullet]] = [L_\bullet]/a_{L_\bullet}$ into the form

$$\Delta([[L_\bullet]]) = \sum_{M_\bullet, N_\bullet} t^{-\langle N_\bullet, M_\bullet\rangle'} w_{M_\bullet, N_\bullet}^{L_\bullet} \frac{a_{M_\bullet} a_{N_\bullet}}{a_{L_\bullet}} \big(K_{\widehat{N_0}} * [[M_\bullet]]\big) \otimes \big([[N_\bullet]] * K_{\widehat{M_1}}\big). \tag{3.8}$$

The proof of the theorem is presented in the next subsection. Since our construction is symmetric with respect to the involution $*$, we also have

Theorem 3.13. *For an abelian category $\mathcal{A}$ satisfying the conditions (a) – (e), the map*

$$I_-^{\mathrm{e}} : \mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A}) \longrightarrow \mathcal{DH}(\mathcal{A}), \qquad [B] \longmapsto F_B,\ K_\beta \longmapsto K_\beta^*$$

defines an embedding of $\mathbb{C}$-coalgebras

$$I_-^{\mathrm{e}} : \big(\mathcal{H}_{\mathrm{tw}}^{\mathrm{e}}(\mathcal{A}), \Delta_{\mathcal{H}(\mathcal{A})}, \epsilon\big) \longrightarrow \big(\mathcal{DH}(\mathcal{A}), \Delta_{\chi_0}, \epsilon\big).$$

3.3. **Proof of Theorem 3.11.** We begin with introducing several lemmas. Assume that $\mathcal{A}$ satisfies the conditions (a) – (e). In the argument we will use the symbols P_A, Q_A introduced in Definition 3.1.

Lemma 3.14. *For an exact sequence*

$$0 \longrightarrow N_\bullet \longrightarrow C_{A_\bullet} \longrightarrow M_\bullet \longrightarrow 0 \tag{3.9}$$

in $\mathcal{E}_0$ with $A \in \mathrm{Obj}(\mathcal{A})$, $M_\bullet$ and $N_\bullet$ are of the form $C_{X_\bullet}$ with some $X \in \mathrm{Obj}(\mathcal{A})$. Moreover, if $w_{M_\bullet, N_\bullet}^{C_{A_\bullet}} \neq 0$ then one can express $M_\bullet = C_{B_\bullet}$, $N_\bullet = C_{D_\bullet}$ and $w_{M_\bullet, N_\bullet}^{C_{A_\bullet}} = g_{C_{B_\bullet}, C_{D_\bullet}}^{C_{A_\bullet}}$.

Proof. Let us express the exact sequence (3.9) in the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] \arrow[d, shift right, "d_1^N"'] & P_A \arrow[r] \arrow[d, shift right, "f_A"'] & M_1 \arrow[r] \arrow[d, shift right, "d_1^M"'] & 0 \\ 0 \arrow[r] & N_0 \arrow[r] \arrow[u, shift right, "d_0^N"'] & Q_A \arrow[r] \arrow[u, shift right, "0"'] & M_0 \arrow[r] \arrow[u, shift right, "d_0^M"'] & 0 \end{tikzcd}$$

Then it is easily seen that $d_0^N = 0$ and $d_0^M = 0$. By the snake lemma we have the long exact sequence

$$0 \longrightarrow \mathrm{Ker}(d_1^N) \longrightarrow 0 \longrightarrow \mathrm{Ker}(d_1^M) \longrightarrow \mathrm{Coker}(d_1^N) \longrightarrow A \longrightarrow \mathrm{Coker}(d_1^M) \longrightarrow 0.$$

Thus $\mathrm{Ker}(d_1^N) = 0$, which with Fact 3.2 implies $N_\bullet = C_{D_\bullet} \oplus K_\bullet$ with $D \in \mathrm{Obj}(\mathcal{A})$ and $H_*(K_\bullet) = 0$. Since $N_\bullet$ is a subcomplex of $C_{A_\bullet}$, $K_\bullet$ appears in the resolution $C_{A_\bullet}$. Since $C_{A_\bullet}$ is minimal, we have $K_\bullet = 0$.

Next recall the condition (2.7) of $\mathcal{E}_0$. If $H_1(M_\bullet) \neq 0$ then $H_0(N_\bullet) = 0$, but since $H_0(N_\bullet) = D$ we have $D = 0$ and $N_\bullet = 0$. Then $M_\bullet = C_{A_\bullet}$ and it is done.

If $H_1(M_\bullet) = 0$ then we have $M_\bullet = C_{B_\bullet} \oplus K_\bullet$ with $B \in \mathrm{Obj}(\mathcal{A})$ and $H_*(K_\bullet) = 0$. Also in this case the minimality of the resolution $C_{A_\bullet}$ implies $K_\bullet = 0$, so we are done. □

Lemma 3.15. *For any $A, B \in \mathrm{Obj}(\mathcal{A})$ we have*

$$a_{C_{A_\bullet}} = a_A \big|\mathrm{Hom}_{\mathcal{A}}(Q_A, P_A)\big|, \qquad \big|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C_{A_\bullet}, C_{B_\bullet})\big| = \big|\mathrm{Hom}_{\mathcal{A}}(Q_A, P_B)\big|\big|\mathrm{Hom}_{\mathcal{A}}(A, B)\big|.$$

Proof. As mentioned in [Br1, Proof of Lemma 4.3], one can easily see that there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Q_A, P_B) \longrightarrow \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C_{A_\bullet}, C_{B_\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, B) \longrightarrow 0,$$

which yields the assertions. □

Lemma 3.16. *For $A, B, D \in \mathrm{Obj}(\mathcal{A})$ we have*

$$g_{C_{B_\bullet}, C_{D_\bullet}}^{C_{A_\bullet}} / g_{B,D}^{A} = t^{2\langle Q_D, P_B\rangle}.$$

Proof. Using the formula (1.2), $\left|\mathrm{Ext}_{\mathcal{A}}(B,D)_A\right| = \left|\mathrm{Ext}_{\mathcal{C}(\mathcal{A})}(C_{B\bullet},C_{D\bullet})_{C_{A\bullet}}\right|$ and Lemma 3.15, we have

$$\begin{aligned}\frac{g^{C_{A\bullet}}_{C_{B\bullet},C_{D\bullet}}}{g^A_{B,D}} &= \frac{\left|\mathrm{Hom}_{\mathcal{A}}(B,D)\right|}{\left|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C_{B\bullet},C_{D\bullet})\right|}\frac{a_{C_{A\bullet}}}{a_A}\frac{a_B}{a_{C_{B\bullet}}}\frac{a_C}{a_{C_{C\bullet}}} \\ &= \frac{1}{\left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_D)\right|}\frac{\left|\mathrm{Hom}_{\mathcal{A}}(Q_A,P_A)\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_B)\right|\left|\mathrm{Hom}_{\mathcal{A}}(Q_D,P_D)\right|}.\end{aligned}$$

Note that the exact sequences $0 \to C_{Di} \to C_{Ai} \to C_{Bi} \to 0$ $(i = 0,1)$ split since C_{Bi} are projective. So $C_{Ai} \cong C_{Di} \oplus C_{Bi}$, which yields

$$\begin{aligned}\frac{g^{C_{A\bullet}}_{C_{B\bullet},C_{D\bullet}}}{g^A_{B,D}} &= \frac{1}{\left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_D)\right|}\frac{\left|\mathrm{Hom}_{\mathcal{A}}(Q_B\oplus Q_C,P_B\oplus P_D)\right|}{\left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_B)\right|\left|\mathrm{Hom}_{\mathcal{A}}(Q_D,P_D)\right|} \\ &= \left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_D)\right|^{-1}\left|\mathrm{Hom}_{\mathcal{A}}(Q_B,P_D)\right|\left|\mathrm{Hom}_{\mathcal{A}}(Q_D,P_B)\right| \\ &= \left|\mathrm{Hom}_{\mathcal{A}}(Q_D,P_B)\right|.\end{aligned}$$

The last equation is equal to $t^{2\langle Q_D,P_B\rangle}$, since by the hereditarity of $\mathcal{A}$ we have

$$t^{2\langle Q_D,P_B\rangle} = q^{\langle Q_D,P_B\rangle} = \left|\mathrm{Hom}_{\mathcal{A}}(Q_D,P_B)\right|/\left|\mathrm{Ext}_{\mathcal{A}}(Q_D,P_B)\right|$$

and we also have $\left|\mathrm{Ext}_{\mathcal{A}}(Q_D,P_B)\right| = 1$ by $P_B \in \mathrm{Obj}(\mathcal{P})$. □

Lemma 3.17. *For $A \in \mathrm{Obj}(\mathcal{A})$ we have*

$$\Delta_{\chi_0}([C_{A\bullet}]) = \sum_{B,D\in\mathrm{Iso}(\mathcal{C}(\mathcal{A}))} t^{\langle D,P_B\rangle-\langle Q_D,B\rangle} g^A_{B,D}\big(K_{\widehat{Q_D}} * [C_{B\bullet}]\big)\otimes\big([C_{D\bullet}] * K_{\widehat{P_B}}\big).$$

Proof. By Lemma 3.14 and the formula (1.2) we have

$$\begin{aligned}\Delta_{\chi_0}([C_{A\bullet}]) &= \sum_{M_\bullet,N_\bullet\in\mathrm{Iso}(\mathcal{C}(\mathcal{P}))} t^{-\langle N_\bullet,M_\bullet\rangle'}\frac{\left|W^{C_{A\bullet}}_{M_\bullet,N_\bullet}\right|}{\left|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M_\bullet,N_\bullet)\right|}\frac{a_{C_{A\bullet}}}{a_{M_\bullet}a_{N_\bullet}}\big(K_{\widehat{N_0}} * [M_\bullet]\big)\otimes\big([N_\bullet] * K_{\widehat{M_1}}\big) \\ &= \sum_{B,D\in\mathrm{Iso}(\mathcal{A})} t^{-\langle C_{D\bullet},C_{B\bullet}\rangle'}\frac{\left|\mathrm{Ext}^1_{\mathcal{C}(\mathcal{A})}(C_{B\bullet},C_{D\bullet})_{C_{A\bullet}}\right|}{\left|\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(C_{B\bullet},C_{D\bullet})\right|}\frac{a_{C_{A\bullet}}}{a_{C_{B\bullet}}a_{C_{D\bullet}}}\big(K_{\widehat{Q_D}} * [C_{B\bullet}]\big)\otimes\big([C_{D\bullet}] * K_{\widehat{P_B}}\big) \\ &= \sum_{B,D} t^{-\langle C_{D\bullet},C_{B\bullet}\rangle'} g^{C_{A\bullet}}_{C_{B\bullet},C_{D\bullet}}\big(K_{\widehat{Q_D}} * [C_{B\bullet}]\big)\otimes\big([C_{D\bullet}] * K_{\widehat{P_B}}\big)\end{aligned}$$

Then by Lemma 3.16 we have

$$\Delta_{\chi_0}([C_{A\bullet}]) = \sum_{B,D} t^{-\langle C_{D\bullet},C_{B\bullet}\rangle'+2\langle Q_D,P_B\rangle} g^A_{B,D}\big(K_{\widehat{Q_D}} * [C_{B\bullet}]\big)\otimes\big([C_{D\bullet}] * K_{\widehat{P_B}}\big).$$

Note that

$$-\langle C_{D\bullet},C_{B\bullet}\rangle' + 2\langle Q_D,P_B\rangle = 2\langle Q_D,P_B\rangle - \langle Q_D,Q_B\rangle - \langle P_D,P_B\rangle = \langle D,P_B\rangle - \langle Q_D,B\rangle$$

by $\widehat{X} = \widehat{Q_X} - \widehat{P_X}$. Thus we have the conclusion. □

Now we start

Proof of Theorem 3.11. By Fact 3.8, it is enough to show

$$I^{\mathrm{e}}_+ \circ \Delta([A]) = \Delta_{\chi_0}(E_A) \tag{3.10}$$

for any $A \in \mathrm{Obj}(\mathcal{A})$. Recalling the definition (3.6) of $\Delta \equiv \Delta_{\mathcal{H}^{\mathrm{e}}_{\mathrm{tw}}(\mathcal{A})}$ on $\mathcal{H}^{\mathrm{e}}_{\mathrm{tw}}(\mathcal{A})$, we have

$$\begin{aligned}\text{LHS of (3.10)} &= I^{\mathrm{e}}_+ \circ \Delta'(a_A[[A]]) = I^{\mathrm{e}}_+\Big(\sum_{B,D} t^{\langle B,D\rangle} g^A_{B,D}\big([B] * K_{\widehat{D}}\big)\otimes[D]\Big) \\ &= \sum_{B,D} t^{\langle B,D\rangle} g^A_{B,D}\big(E_B * K_{\widehat{D}}\big)\otimes E_D.\end{aligned}$$

Using the definition (3.3) of E_A, we have

$$\text{LHS of (3.10)} = \sum_{B,D} t^{\langle B,D\rangle} g^A_{B,D}\big(t^{\langle P_B,B\rangle}K_{-\widehat{P_B}} * [C_{B\bullet}] * K_{\widehat{D}}\big)\otimes\big(t^{\langle P_D,D\rangle}K_{-\widehat{P_D}} * [C_{D\bullet}]\big).$$

Recalling the relation (1.8) and $\widehat{C_{X\bullet}} = \widehat{X}$, we have

$$\text{LHS of (3.10)} = \sum_{B,D} t^{\langle B,D\rangle+\langle P_B,B\rangle+\langle P_D,D\rangle+(-\widehat{P_D},\widehat{C_{D\bullet}})-(\widehat{D},\widehat{C_{B\bullet}})} g^A_{B,D}\big(K_{-\widehat{P_B}} * K_{\widehat{D}} * [C_{B\bullet}]\big)\otimes\big([C_{D\bullet}] * K_{-\widehat{P_D}}\big)$$

$$= \sum_{B,D} t^{\langle B,D\rangle+\langle P_B,B\rangle+\langle P_D,D\rangle-(P_D,D)-(D,B)} g^A_{B,D}\big(K_{-\widehat{P_B}} * K_{\widehat{D}} * [C_{B\bullet}]\big) \otimes \big([C_{D\bullet}] * K_{-\widehat{P_D}}\big)$$
$$= \sum_{B,D} t^{\langle P_B,B\rangle-\langle D,P_D\rangle-\langle D,B\rangle} g^A_{B,D}\big(K_{-\widehat{P_B}} * K_{\widehat{D}} * [C_{B\bullet}]\big) \otimes \big([C_{D\bullet}] * K_{-\widehat{P_D}}\big). \tag{3.11}$$

On the other hand, using Lemma 3.17 we have

$$\begin{aligned}\text{RHS of (3.10)} &= \Delta_\chi(t^{\langle P_A,A\rangle} K_{-\widehat{P_A}} * [C_{A\bullet}]) \\ &= t^{\langle P_A,A\rangle} \sum_{B,D} t^{\langle D,P_B\rangle-\langle Q_D,B\rangle} g^A_{B,D}\big(K_{-\widehat{P_A}} * K_{\widehat{Q_D}} * [C_{B\bullet}]\big) \otimes \big(K_{-\widehat{P_A}} * [C_{D\bullet}] * K_{\widehat{P_B}}\big).\end{aligned}$$

Using the formula (1.8) we have

$$\begin{aligned}&\text{RHS of (3.10)} \\ &= \sum_{B,D} t^{\langle P_A,A\rangle+\langle D,P_B\rangle-\langle Q_D,B\rangle+(-\widehat{P_A},\widehat{C_{D\bullet}})} g^A_{B,D}\big(K_{-\widehat{P_A}} * K_{\widehat{Q_D}} * [C_{B\bullet}]\big) \otimes \big([C_{D\bullet}] * K_{-\widehat{P_A}} * K_{\widehat{P_B}}\big) \\ &= \sum_{B,D} t^{\langle P_A,A\rangle+\langle D,P_B\rangle-\langle Q_D,B\rangle-(P_A,D)} g^A_{B,D}\big(K_{-\widehat{P_A}} * K_{\widehat{Q_D}} * [C_{B\bullet}]\big) \otimes \big([C_{D\bullet}] * K_{-\widehat{P_A}} * K_{\widehat{P_B}}\big).\end{aligned} \tag{3.12}$$

Now we compare (3.11) and (3.12). We can assume $\widehat{A} = \widehat{B} + \widehat{D}$ since otherwise $g^A_{B,D} = 0$. Since we are considering the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P_B \arrow[r] \arrow[d, shift right, "f_B"'] & P_A \arrow[r] \arrow[d, shift right, "f_A"'] & P_D \arrow[r] \arrow[d, shift right, "f_D"'] & 0 , \\ 0 \arrow[r] & Q_B \arrow[r] \arrow[u, shift right, "0"'] & Q_A \arrow[r] \arrow[u, shift right, "0"'] & Q_D \arrow[r] \arrow[u, shift right, "0"'] & 0 \end{tikzcd}$$

we also have $\widehat{P_A} = \widehat{P_B} + \widehat{P_D}$ and $\widehat{Q_A} = \widehat{Q_B} + \widehat{Q_D}$. Using these relations we easily see

$$\begin{aligned} K_{-\widehat{P_B}} * K_{\widehat{D}} &= K_{-\widehat{P_A}} * K_{\widehat{Q_D}}, \\ K_{-\widehat{P_A}} * K_{\widehat{P_B}} &= K_{-\widehat{P_D}}. \end{aligned}$$

Moreover we can compute

$$\begin{aligned}\langle P_A, A\rangle + \langle D, P_B\rangle - \langle Q_D, B\rangle - (P_A, D) &= \langle P_A, B\rangle + \langle P_A, D\rangle + \langle D, P_B\rangle - \langle D, B\rangle - \langle P_D, B\rangle - (P_A, D) \\ &= \langle P_A, B\rangle + \langle D, P_B\rangle - \langle D, B\rangle - \langle P_D, B\rangle - \langle D, P_A\rangle \\ &= \langle P_B, B\rangle + \langle D, P_B\rangle - \langle D, B\rangle - \langle D, P_B\rangle - \langle D, P_D\rangle \\ &= \langle P_B, B\rangle - \langle D, P_D\rangle - \langle D, B\rangle.\end{aligned}$$

Therefore (3.11) and (3.12) coincide. □

3.4. **Bialgebra structure of Bridgeland's Hall algebra.** We continue to consider the product on $\mathcal{DH}(\mathcal{A}) \widehat{\otimes} \mathcal{DH}(\mathcal{A})$ defined as

$$\big([M_\bullet] \otimes [N_\bullet]\big) * \big([P_\bullet] \otimes [Q_\bullet]\big) := \big([M_\bullet] * [N_\bullet]\big) \otimes \big([P_\bullet] * [Q_\bullet]\big) \tag{3.13}$$

and the map

$$\chi_0(M_\bullet, N_\bullet) := -\langle N_\bullet, M_\bullet\rangle.$$

Theorem 3.18. *For an abelian category $\mathcal{A}$ satisfying the conditions (a)–(e), the tuple $(\mathcal{DH}(\mathcal{A}), *, [0], \Delta_{\chi_0}, \epsilon)$ is a topological bialgebra under the multiplication* (3.13) *on $\mathcal{DH}(\mathcal{A}) \widehat{\otimes} \mathcal{DH}(\mathcal{A})$.*

Proof. By the construction of Δ_{χ_0} (Definition 3.9), it is enough to show $\Delta_{\chi_0}(E_A * E_B) = \Delta_{\chi_0}(E_A) * \Delta_{\chi_0}(E_B)$. But it follows from Fact 3.8 and Theorem 3.11, so we are done. □

Combining the result of [Go] and [Ya], we also have

Theorem 3.19. *For an abelian category $\mathcal{A}$ satisfying the conditions (a)–(e), the tuple $(\mathcal{DH}_{\mathrm{red}}(\mathcal{A}), *, [0], \Delta_{\chi_0}, \epsilon)$ coincides with the Drinfeld double of $(\mathcal{H}^{\mathrm{e}}_{\mathrm{tw}}(\mathcal{A}), *, [0], \Delta_{\mathcal{H}^{\mathrm{e}}_{\mathrm{tw}}(\mathcal{A})}, \epsilon)$ as a bialgebra.*

References

[Br1] Bridgeland, T., *Quantum groups via Hall algebras of complexes*, Ann. of Math. (2) **177** (2013), no. 2, 739–759.

[Br2] Bridgeland, T., *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[1] Cramer, T., *Double Hall algebras and derived equivalences*, Adv. Math. **224** (2010), no. 3, 1097–1120.

[Dr] Drinfeld, V. G., *Quantum groups*, Proceedings of the International Congress of Mathematicians (Berkeley, 1986), 798–820, Amer. Math. Soc. (1987).

[Gr] Green, J., *Hall algebras, hereditary algebras and quantum groups*, Invent. Math. **120** (1995), no. 2, 361–377.

[Go] Gorsky. M., *Semi-derived Hall algebras*, arXiv:1303.5879.

[Hu] Hubery, A., *From triangulated categories to Lie algebras: a theorem of Peng and Xiao*, Contemp. Math. **406** (2006), 51–66.

[Ke] Keller, B., *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417.

[Qu] Quillen, D., *Higher algebraic K-theory. I.*, Lecture Notes in Math., **341** (1973), 85–147.

[Ri] Ringel, C., *Hall algebras and quantum groups*, Invent. Math. **101** (1990), no. 3, 583–591.

[R2] Ringel, C., *Green's theorem on Hall algebras*, Representation theory of algebras and related topics (Mexico City, 1994), 185–245, CMS Conf. Proc., **19**, Amer. Math. Soc., 1996.

[Sc] Schiffmann, O., *Lectures on Hall algebras*, arXiv:0611617v2.

[2] Xiao, J., *Drinfeld double and Ringel-Green theory of Hall algebras*, J. Algebra **190** (1997), no. 1, 100–144.

[Ya] Yanagida, S., *A note on Bridgeland's Hall algebra of two-periodic complexes*, arXiv:1207.0905.

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